Indian J. Phys. **72A** (5), 439–446 (1998)

IJP A
— an international journal

Metastability and hysteresis in random field Ising chains

Prabodh Shukla

Department of Physics, North Eastern Hill University,
Shillong-793 022, India

Abstract : Zero-temperature non-equilibrium dynamics of one dimensional random field Ising models is analysed for metastable states and disorder-driven hysteresis. Ferromagnetic as well as anti-ferromagnetic cases are considered. In the ferromagnetic case, we obtain an exact expression for the hysteresis loop in the zero-frequency limit. In the anti-ferromagnetic case, an exact solution of the problem is not possible (so far). Some interesting aspects of the anti-ferromagnetic dynamics are discussed. Its relationship with the dynamics of an ANNNI chain is also examined.

Keywords : Ising model, metastable states, disorder-driven hysteresis

PACS Nos. : 75.60.E, 75 70.K

1. Introduction

The random field Ising model (RFIM) has played an important role in understanding disordered systems. It first came into prominence around 1975, when Imry and Ma [1] argued that Ising magnets with a quenched random field were incapable of sustaining a long range order below two dimensions. It was an appealing argument, and a kind of a clear statement which was lacking in the context of other prominent models of quenched disorder, *e.g.* the Sherrington-Kirkpatric (random bond) model of a spinglass [2]. Thus several people were attracted to the study of the equilibrium statistical mechanics of the random field Ising model. Soon a controversy was generated. Dimensional reduction arguments based on field theoretic methods showed that the lower critical dimensionality of RFIM was three rather than two as predicted by Imry and Ma. It took some years to resolve that the application of the dimensional reduction method in this context was unjustified because it necessarily assumed the existence of a unique solution of the field equations. It was shown that the field equations for systems with quenched disorder have a large number

of solutions (metastable states). Due to the presence of numerous metastable states in the system, the numerical simulations too proved rather difficult and inconclusive, and the initial enthusiasm for the model faded in due course.

Some years later, interest in RFIM revived for the same reason it had faded earlier. Its richness in metastable states was a deterent in the study of its equilibrium properties, but made it a good model for the study of nonequilibrium phenomena in glassy and complex systems. These systems are characterised by extremely slow relaxation, and history dependent effects which arise from the presence of several metastable states in the system. There are two broad time scales; (i) the lifetime of the metastable state (τ_1), and the transition time between neighbouring metastable states (τ_2). Generally, $\tau_2 \ll \tau_1$, and thermal excitations are too weak to push the system from one metastable state to another over practical time scales. However, a sufficiently strong external force can easily achieve this. A cyclic driving force takes the system through a hysteresis loop. The loop shows that the system can rest in two different states for the same value of the external parameters depending upon the history of the system. This is a non-equilibrium effect, and one can argue that in the limit of the frequency of the driving force going to zero, the area of the loop will also go to zero. This is fine, but in several systems hysteresis loops are observed even at driving frequencies of the order of 10^{-6} Hz (corresponding to periods of several days). These loops show no sign of disappearing over time scales which test the patience of the experimentalist. Thus, for practical purposes, we need a theory for this nonequilibrium phenomena.

Recently, Sethna *et al* [3] used the RFIM to study hysteresis and other related phenomena such as the return point memory effect, and the Barkhausen noise. Hysteresis is a kinetic phenomenon, and therefore one needs to put in a dynamics in the model. Sethna *et al* employed the zero-temperature Glauber dynamics of Ising spins. It showed remarkable success in reproducing the observed features of hysteresis and other phenomena mentioned above. The success of the Sethna model is not unreasonable. It is a minimal model which takes into account the most important aspects of hysteresis. The zero-temperature dynamics effectively sets $\tau_1 = \infty$, and $\tau_2 = 0$. This is a reasonable approximation at finite temperatures on laboratory time scales. Although the dynamics is deterministic, there is a stochastic aspect to it coming from the randomness of the quenched field. The metastable states of the RFIM become fixed points (stable states) under the zero-temperature dynamics. This simplifies their numerical as well as analytic characterization. However, the model retains the key features of the original problem. There is a broad distribution of energy barriers between nearby stable states. When the system is driven by a smoothly increasing applied field, it jumps from a stable state to a nearby stable state of lower energy when the applied field crosses the barrier between the two states. As the barriers are random variables, the trajectory of the system is not smooth. On a microscopic scale, it consists of irregular jumps in the magnetization (Barkhausen noise).

Experiments show that there is a broad distribution of the size of the magnetization jumps. Averaged over the entire hysteresis loop, jump distribution shows power laws over

several decades (usually three). This has lead to suggestions that there is a self-organized criticality in the system. The Sethna model provides a framework for examining this question. Although it does not appear to support self-organised criticality in the system, but there is a "plain old critical point" on each half of the hysteresis loop. At this point, the magnetization jumps show true power laws. The critical region appears to be rather broad. Thus approximate power laws are expected over a wide sector of the hysteresis loop. The extensive study of the Sethna model is based on numerical simulations of the model, and its analysis in the mean field approximation. We have initiated a modest effort to solve the Sethna model exactly in one dimension, and also on Bethe lattices to clarify its critical behaviour. Here, we limit ourselves to the one dimensional case. Although one dimension is definitely below the lower critical dimension of the random Ising model, but the model shows interesting and non-trivial non-equilibrium phenomena. In fact there is nothing very one dimensional about the hysteresis loop in the one dimensional model. It looks qualitatively similar to the one in three dimensions. The analysis of the one dimensional model serves to illustrate the basic method which can be applied to Bethe lattices as well. As we shall see below, there are several questions which cannot be answered (so far) even in the one dimensional case. We hope that readers may be persuaded to investigate these questions.

2. The model

Consider a one dimensional Ising model with spins $\{s_i = \pm 1\}$, nearest neighbour interaction J, a uniform external field h, and a quenched random field h_i at each site i drawn from a continuous probability distribution $p(h_i)$.

The effective field seen by a spin s_i is given by :

$$l_i = J(s_{i-1} + s_{i+1}) + h_i + h \tag{1}$$

The energy of spin s_i is equal to $-l_i s_i$. The zero-temperature relaxational dynamics of the system attempts to lower the total energy of the system by flipping each spin which is not aligned in the direction of the local field at its site. It updates spins according to the rule,

$$s_i = \text{sign}(l_i) \tag{2}$$

The relaxational dynamics is an iterative process. If lowering the energy of a spin increases the energy of one of its neighbours, then that neighbour is updated at the next step. After a number of steps, the dynamics converges to a stable configuration where each spin satisfies equation (2).

The total energy of the system is given by,

$$H = -J\sum_{i,j} s_i s_j - \sum_i h_i s_i - h\sum_i s_i \tag{3}$$

A state satisfying equation (2) is a local minimum of the energy of the system. It may be possible to obtain states of lower energy by flipping pairs or larger clusters of spins together, but these states are outside the scope of the dynamics considered here.

The locally stable state obtained by our dynamics depends on the history of the system. For example, two initial states, one with all spins down, and the other with all spins up yield different stable states at the same applied field h. We focus on the lower half of the hysteresis loop. In the following, we outline a method to calculate the magnetization per spin $m(h)$ in a field h starting from a saturated state ($m = -1$) at $h = -\infty$. The magnetization in the upper half is related to $m(h)$ by the symmetry $m(h) = -m(-h)$. At present we are not able to calculate the magnetization $m(h)$ for an arbitrary initial state.

3. Ferromagnetic interactions

The model described above possesses two important properties if the nearest neighbour interactions are ferromagnetic ($J > 0$). These properties are :

1. The stable state does not depend upon the trajectory of the applied field from $h = -\infty$ to h, as long as it remains everywhere bounded below h.
2. The stable state does not depend upon the order in which the spins are updated during the relaxational process.

The above properties greatly simplify the analysis of the model. Suppose we wish to calculate $m(h)$ starting from $m(-\infty) = -1$. In view of the first property, we do not have to worry about the detailed trajectory of the applied field if it was raised from $h = -\infty$ to its present value h slowly. We can start with the initial state with all spins down, and relax it directly in field h.

To calculate $m(h)$, we have to calculate the probability that a spin at an arbitrary lattice site O is up in the relaxed state at field h. This calculation is performed in two steps.

In the first step, the spin at the site O is kept down, but all other spins on the lattice are relaxed. The spin at O is connected to two semi-infinite lattices, and spins on each half-lattice can be relaxed independently of the other half-lattice. We focus on one half-lattice, say the one on the left of O. Consider a long chain of N spins extending to the left of the site O. Number its sites by $n = 1, 2, \ldots, N-1, N, N+1$; $n = N+1$ denoting the site O. We relax the spins on this chain in the following order. Spin at site 1 is relaxed first, then at site 2, and so on. Relaxing a spin means checking the local field on that spin, and if it is positive, to turn the spin up. The spin at site $n = 1$ has only one neighbour which is necessarily down (because it is not relaxed so far). Thus the local field at the end spin is $l_1 = -J + h_1 + h$, and we turn it up if l_1 is positive. Next we relax the spin at site $n = 2$ but keeping the spin at $n = 3$ down. If spin at site 2 turns up during the relaxation, we re-examine site 1 to see if it would turn up as well. Similarly if a spin at site (n) turns up, spin at site ($n - 1$) is re-examined, if this turns up then the spin at ($n - 2$) is re-examined, and so on till we come to a site where the spin is either already up, or it is down and remains down even after its right neighbour has turned up.

The advantage of choosing the above order for relaxing the spins is that we can write a recursion relation for the probability that a spin is up at site (n), given that the spin at site ($n + 1$) is down. This probability becomes independent of n if $n \gg 1$, *i.e.* if one is sufficiently

far from the end of the chain. Let $P^n(h)$ be the conditional probability that a randomly chosen spin at site n is up, given that its nearest neighbour at site $n+1$ is down (not relaxed yet), but the spin itself and all spins to its left are relaxed. We obtain,

$$P^n(h) = p_1(h)P^{n-1}(h) + p_0(h)[1 - P^{n-1}(h)] \tag{4}$$

Here $p_m(h)$ is the probability that the local field at a site is positive if m of its nearest neighbors are up, ($m = 0, 1, 2$).

$$p_m(h) = \int_{2(1-m)J-h}^{\infty} p(h_i)dh_i \tag{5}$$

The probability that the local field on a boundary site is positive is

$$P^1(h) = \int_{J-h}^{\infty} p(h_i)dh_i \tag{6}$$

Using the above initial condition we can determine all P^n recursively for $n \geq 1$. For large n, $P^n(h)$ tends to a fixed point given by the self consistent equation,

$$P^*(h) = p_1(h)P^*(h) + p_0(h)[1 - P^*(h)] \tag{7}$$

The second step is to relax the spin at site O. Its two nearest neighbors have been relaxed, and each of these is up independently with probability $P^*(h)$. Thus the probability that the spin at site O is up is given by,

$$p(h) = p_2(h)[P^*]^2 + 2p_1(h)P^*[1 - P^*] + p_0[1 - P^*]^2 \tag{8}$$

We obtain,

$$p(h) = p_0 \frac{1 - p_1^2 + p_0 p_2}{1 - (p_1 - p_0)} \tag{9}$$

The magnetization per spin (on the lower hysteresis loop) is given by

$$m(h) = 2p(h) - 1 \tag{10}$$

The above results were derived in reference [4] by an alternate method, and checked numerically by Monte Carlo simulations. These results have been extended to Bethe lattices as well [5]. Somewhat surprisingly, the behaviour on a Bethe lattices with coordination number three is similar to the one dimensional case, but behavior on lattices of higher coordination number is qualitatively different. We refer the reader to reference [5] for details.

4. Anti-ferromagnetic interactions

The anti-ferromagnetic chain is described by a negative $J(J < 0)$. In this case the two properties of the ferromagnetic model listed at the beginning of the preceding section are lost. Therefore the method developed there is no longer useful. We describe briefly a typical numerical simulation. For simplicity, consider a flat and bounded distribution of quenched fields in the range $-\Delta \leq h_i \leq \Delta$. Start with a sufficiently large and negative applied field such that all spins are down $\{s_i = -1\}$ initially, and raise the field slowly. The anti-

ferromagnetic interaction does not like the adjacent spins to be aligned in parallel, and therefore the applied field has to be more negative than $2J - \Delta$ to keep all the spins down. At, $h = 2J - \Delta$, the first spin flips up. On raising the applied field further, more spins flip up, and the magnetization rises to a value equal to $-\exp(-2)$ at $h = 2J + \Delta$. The magnetization remains fixed at this value (first plateau) upto $h = -\Delta$. Further increase in the applied field from $-\Delta$ to Δ, increases the magnetization continuously to a value which is about 10% lower than $\exp(-2)$. It remains fixed at this value (second plateau) upto $h = -2J - \Delta$. Further increase in the applied field cause the remaining spins to turn up gradually, and at $h = -2J + \Delta$, we get $m = 1$. This completes the lower half of the hysteresis loop in an increasing applied field. The upper half loop lies very close to the lower half, and therefore the area of the hysteresis loop is very small. These features of the zero-temperature anti-ferromagnetic dynamics are easy to understand. Limitation of space does not allow us to go into the details here. We refer the reader to reference [6] for an approximate analysis of the numerical results. An exact analysis has not been possible so far, but we are working on it.

The key to understand the anti-ferromagnetic dynamics is to note that when a spin flips up, an adjacent spin, if it was down initially, is stabilized in its down position. Therefore a spin flipping up as a result of increased applied field does not give rise to the possibility of an avalanche. It may cause a neighboring up spin to flip back down, but it can not change the state of spins beyond the nearest neighbor. In other words, a microscopic increment in the applied field never causes more than two spins to flip. The two-flips are relatively uncommon (less than 4% approximately). In the majority of cases spins are turned up one at a time. The smallness of the two-flip effect is responsible for the smallness of the hysteresis in the simulations.

5. Random ANNNI chain

ANNNI (axial next nearest neighbor Ising) chain [7] is described by the hamiltonian,

$$H_1 = -J_1 \sum_i s_i s_{i+1} - J_2 \sum_i s_i s_{i+2}, \tag{11}$$

Here J_1 and J_2 are competing interactions ($J_1 > 0$, and $J_2 < 0$, or $J_1 < 0$, and $J_2 < 0$). It supports a rich and complex short-range structure, and has been studied extensively in the context of spatially modulated periodic structures in magnetic and other systems. The phase diagrams of the ANNNI chain obtained from a dynamic criterion often show considerable differences from those obtained from purely energetic considerations. Numerical studies indicate that the most stable states of the system (the true equilibrium states) are not necessarily the most probable states of the system. Some issues in this context can be clarified by the study mentioned in the preceding section. Defining new Ising spins $\sigma_i = s_i s_{i+1}$, H_1 can be transformed into the form

$$H_2 = -J_1 \sum_i \sigma_i - J_2 \sum_i \sigma_i \sigma_{i+1} \tag{12}$$

Hamiltonian H_2 is similar to the one studied in the previous section, and gives us an occasion to comment on the effects of quenched randomness on the non-equilibrium

dynamics of the ANNNI chain. Let $J_1 = \bar{J}_1 + h_i$ where h_i is a quenched random variable with zero mean value. We can make contact with the equilibrium states of the non-random chain in the limit $\Delta \to 0$. An equilibrium state is determined by energetic considerations alone; it is the global minimum of energy. The dynamically stable states (which may correspond to the meta-stable states at finite temperatures) are the local minima of energy. We wish to compare the nature of ordering in the two sets of the states.

It is useful to recall the equilibrium results for the non-random ANNNI chain. Consider Hamiltonian H_1 with $J_1 < 0$, and $J_2 < 0$. The zero-temperature ground state is ferromagnetic if $J_1 < 2J_2$, and an anti-phase state (two spins up followed by two spins down and so on) if $J_1 > 2J_2$. The anti-phase state can be seen easily with the help of the transformed Hamiltonian H_2. For $J_1 = 2J_2$, the ground state is not very discriminating with respect to any particular long-range order. It is infinitely degenerate with any sequence of k-bands (k adjacent identically oriented spins, terminated at both ends by oppositely oriented spins) having the same energy. The degeneracy goes up as a Fibonacci series, and scales as d^N, where N is the number of spins in the chain, and $d = (\sqrt{5}+1)/2$. The two spin correlation function $\langle s_i s_{i+r} \rangle$, averaged over the degenerate states can be obtained analytically, and decays exponentially with an oscillatory modulation. At finite temperatures, there are two qualitatively different regimes. For $J_1 < 2\kappa J_2$, where κ is a temperature-dependent parameter, the correlations decay exponentially without an oscillatory modulation. For $J_1 > 2\kappa J_2$, the exponential decay of correlations is spatially modulated by a multiplicative factor of the form cos qr where q varies with κ as well as temperature.

Coming to the random ANNNI chain, we see that in the region $J_1 < 2J_2 - \Delta$ (region A), the dynamically stable state is a ferromagnetic state with all σ spins down. In the region $2J_2 - \Delta \le J_1 \le -2J_2 + \Delta$ (region B), the system settles into a state of an arbitrary sequence of k-bands. The number and the structure of the dynamically stable states in region B is the same as that of the equilibrium states of the non-random chain at $J_1 = 2J_2$. In the region, $2J_2 - \Delta \le J_1 \le -\Delta$ (region C), the dynamically stable states are the jammed states (the states on the plateaus mentioned in the preceding section). The jammed states occur over a large region (Δ can be arbitrarily small), and have a certain universality in the sense that they can be characterized by a common property (no more than two consecutive spins are parallel) independently of the parameters of the system over a wide range of the parameters.

In regions A and B, the dynamically stable states have the same structure as the equilibrium states. In region C, however, the dynamics leads to jammed states, while energetic considerations yield the anti-phase state with perfect long-range order. Thus the non-equilibrium dynamical effects are most striking in region C. The jammed states have a random distribution of energies, but are statistically similar in structure. The structure factor of these "glassy" states can be calculated analytically. There is no true long range order in the jammed states, but large sections of the jammed chain can show periodic structures which are quite similar to the anti-phase state.

6. Concluding remarks

There is an obvious scarcity of exact results in the field of random systems and non-equilibrium statistical mechanics. We have described a method which provides an exact result in one dimension for the zero-temperature non-equilibrium dynamics of the random field Ising model with ferromagnetic interactions. The method can be generalised to a Bethe lattice. Hysteresis loops as well as avalanche distributions (Barkhausen noise) can be obtained exactly [8]. So far we have been unable to solve the problem of anti-ferromagnets exactly, but work is in progress in this direction.

References

[1] Y Imry and S K Ma *Phys. Rev. Lett.* **35** 1399 (1975); For a more modern reference, see T Nattermann, cond-mat 9705259

[2] D Sherrington and S Kirkpatrick *Phys. Rev. Lett.* **35** 1972 (1975)

[3] J P Sethna, K Dahmen, S Kartha, J A Krumhansl, B W Roberts and J D Shore *Phys. Rev. Lett.* **70** 3347 (1993); O Perkovic, K Dahmen and J P Sethna *Phys. Rev. Lett.* **75** 4528 (1995)

[4] *Exact solution of zero-temperature hysteresis in a ferromagnetic Ising chain with quenched random fields* P Shukla *Physica* **A233** 235 (1996)

[5] *Zero-temperature hysteresis in the random field Ising model on a Bethe lattice* Deepak Dhar, Prabodh Shukla and James P Sethna *J. Phys.* **A30** 5259 (1997)

[6] *Zero-temperature hysteresis in an anti-ferromagnetic Ising chain with quenched random fields* P Shukla *Physica* **A233** 242 (1996)

[7] W Selke *Physics Reports* **170** 213 (1988); See also, W Selke *in Phase Transitions and Critical Phenomena* **Vol 15** eds C Domb and J L Lebowitz (London · Academic Press) and references therein (1992)

[8] To be published (collaborative work)